J. Fluid Mech. (1977), *vol.* 80, *part* 1, *pp.* 149–152
Printed in Great Britain

Note on a solitary wave in a slowly varying channel

By JOHN W. MILES

Institute of Geophysics and Planetary Physics,
University of California, La Jolla

(Received 6 October 1976)

Johnson's (1973) description of a solitary wave in water of slowly varying depth is extended to a channel of slowly varying breadth and depth b and d on the assumption that the scale for the variation of b and d is large compared with $d^{\frac{3}{2}}/a^{\frac{1}{2}}$. It is inferred from conservation of energy that the amplitude of the wave is proportional to $b^{-\frac{2}{3}}d^{-1}$ (cf. Green's law $a \propto b^{-\frac{1}{2}}d^{-\frac{1}{4}}$ for long waves of small amplitude). Comparison with experiment (Perroud 1957) yields fairly satisfactory agreement for a linearly converging channel of constant depth. The agreement for a linearly diverging channel is not satisfactory, but the experimental data are inadequate to support any firm conclusion.

A solitary wave of amplitude a_0 in water of uniform depth d_0 may be described by (Lamb 1932, § 252)

$$\eta(x,t) = a_0 \operatorname{sech}^2\left(\frac{x-ct}{l_0}\right), \quad c = \{g(d_0+a_0)\}^{\frac{1}{2}} \equiv c_0(1+\alpha)^{\frac{1}{2}}, \tag{1a, b}$$

where

$$l_0 = 2(d_0^3/3a_0)^{\frac{1}{2}}, \quad c_0 = (gd_0)^{\frac{1}{2}}, \tag{2a, b}$$

$$\alpha = a_0/d_0 \ll 1, \tag{3}$$

and error factors of $1+O(\alpha, \alpha^2)$ are implicit in ($1a, b$). The form of (1), together with Green's analysis of the corresponding linear problem (Lamb 1932, § 185), suggests that a solitary wave in a channel of slowly varying breadth and depth $b(x)$ and $d(x)$ may be described by

$$\eta(x,t) = a \operatorname{sech}^2\left\{\frac{(3ga)^{\frac{1}{2}}}{2d}\left(\int_0^x \frac{dx}{c} - t\right)\right\}, \quad c = \{g(d+a)^{\frac{1}{2}}\}, \tag{4a, b}$$

where $a(x)$ is a slowly varying amplitude.

The validity of (4) for a channel of constant breadth follows directly from Johnson's (1973) asymptotic analysis, which also implies that: the error factor for ($4a$) is $1+O(\alpha, \lambda/\alpha)$, where

$$\lambda \equiv l_0/l_1 \ll \alpha, \tag{5}$$

l_1 is the scale of the slow variation, and l_0 is given by ($2a$) with a_0 and d_0 as reference values of a and d; the approximation is not uniformly valid for either $x \to \infty$ or $t \to \infty$ (so that the wave departs significantly from the sech² profile in the region of small displacement); a must be inversely proportional to d in consequence of the matching requirement between (4), *qua* inner approximation, and the corresponding outer approximation.

The error in neglecting the transverse variation of η, which arises from the requirement that the ratio of the transverse velocity to the axial velocity at the side walls must be $\pm db/dx$, may be estimated by considering oblique reflexion of a solitary wave at a plane wall. Regular reflexion occurs if and only if $\theta_i > (3\alpha)^{\frac{1}{2}}$, where θ_i is the angle of incidence (Miles 1977*a*). Mach reflexion occurs if $\theta_i < (3\alpha)^{\frac{1}{2}}$, and the relative strength of the reflected wave is $\theta_i/(3\alpha)^{\frac{1}{2}}$ (Miles 1977*b*). This, together with the closely related result that a solitary wave moving along a wall cannot be turned through a convex angle greater than $(3\alpha)^{\frac{1}{2}}$, implies the restriction

$$|db/dx| \leqslant (3\alpha)^{\frac{1}{2}}, \tag{6}$$

which is typically weaker than the restrictions, especially (5), already implicit in the asymptotic approximation; accordingly (4) remains valid for a channel of slowly varying breadth.

Johnson's analysis may be generalized to establish the variation of a with b; however, the desired result may be inferred more directly from conservation of energy, which implies the conservation of a^2bl for a wave of amplitude a and length l in a channel of width b [cf. Rayleigh's derivation of Green's law (Lamb 1932, § 185)]. Combining this result with $l \propto d^{\frac{3}{2}}/a$ (see above) then implies the conservation of $(ad)^{\frac{3}{2}}b$, from which we infer that

$$a/a_0 = (b/b_0)^{-\frac{2}{3}}(d/d_0)^{-1}, \tag{7}$$

where a_0, b_0 and d_0 are reference values.

We remark that (7) implies

$$\frac{(3ga)^{\frac{1}{2}}}{2d} = \frac{c_0}{l_0}\left(\frac{b}{b_0}\right)^{-\frac{1}{3}}\left(\frac{d}{d_0}\right)^{-\frac{3}{2}} \tag{8}$$

in (4*a*), so that

$$\eta(x,t) = a\,\mathrm{sech}^2\left\{\frac{c_0}{l_0}\left(\int_0^x \frac{dx}{c} - t\right)\right\}, \quad \frac{a}{a_0} = \left(\frac{d}{d_0}\right)^2 \tag{9a, b}$$

if $bd^{\frac{9}{2}}$ is constant. Asymptotic analysis reveals that, in this rather curious case, the error factor is $1 + O(\alpha, \lambda)$ and that the approximation is uniformly valid with respect to both x and t.

We conclude that the local speed of a solitary wave in a slowly varying channel is simply $\{g(d+a)\}^{\frac{1}{2}}$ and that Green's law for long waves of small amplitude, $a \propto b^{-\frac{1}{2}}d^{-\frac{1}{4}}$ (Lamb 1932, § 185), is replaced by $a \propto b^{-\frac{2}{3}}d^{-1}$ if the scale for the variation of breadth and depth, say l_1, is large compared with $d^{\frac{5}{2}}/a^{\frac{3}{2}}$. It may be, as often happens for approximations of Green's type, that (6) has a greater range of validity than the restriction (5) appears to suggest, but it must not be overlooked that a solitary wave advancing into water of *decreasing* depth may undergo fission if l_1 is comparable with $d_0/\alpha^{\frac{3}{2}}$ (see Madsen & Mei 1969; Johnson 1973).

Comparison with experiment

Perroud (1957) measured the amplitude variation of a solitary wave in two linearly converging channels and one linearly diverging channel of constant

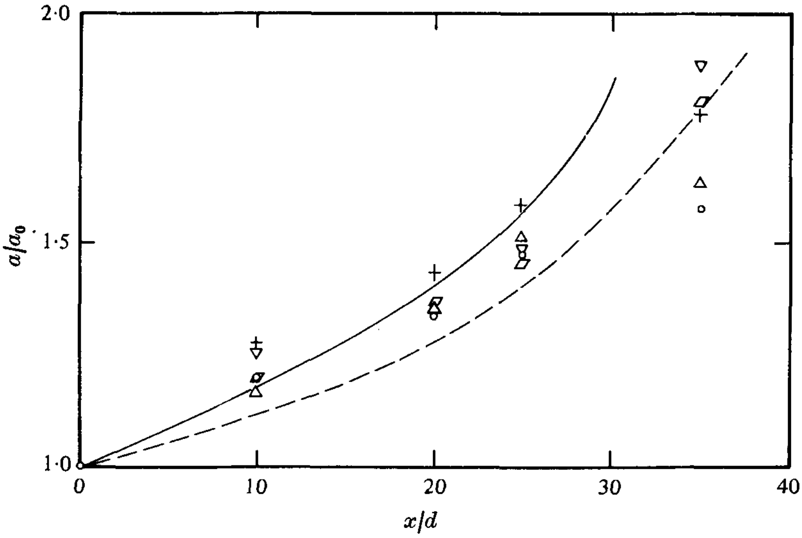

FIGURE 1. Amplitude of a solitary wave in a converging channel for which b/d decreases linearly from 5·2 to 0 as x/d increases from 0 to 50. ——, $a/a_0 = (b/b_0)^{-\frac{2}{3}}$; – – –, $a/a_0 = (b/b_0)^{-\frac{1}{2}}$. Observed data: $+$, $\alpha \equiv a_0/d = 0{\cdot}18$; ∇, $\alpha = 0{\cdot}24$; $\square$, $\alpha = 0{\cdot}35$; Δ, $\alpha = 0{\cdot}47$; $\bigcirc$, $\alpha = 0{\cdot}50$. The data for $x/d = 35$ appear to have been significantly affected by partial breaking.

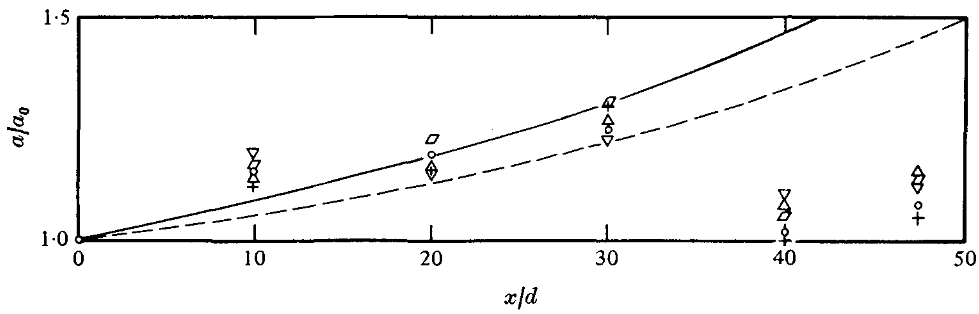

FIGURE 2. Amplitude of a solitary wave in a converging channel for which b/d decreases linearly from 3·75 to 1·7 as x/d increases from 0 to 50. ——, $a/a_0 = (b/b_0)^{-\frac{2}{3}}$; – – –, $a/a_0 = (b/b_0)^{-\frac{1}{2}}$. Observed data: $+$, $\alpha = a_0/d = 0{\cdot}15$; ∇, $\alpha = 0{\cdot}20$; $\square$, $\alpha = 0{\cdot}35$; Δ, $\alpha = 0{\cdot}45$; $\bigcirc$, $\alpha = 0{\cdot}50$. The data for $x/d = 40$ and 47 appear to have been significantly affected by wall friction.

depth and compared his results with Green's law, $a \propto b^{-\frac{1}{2}}$. (He also compared them with $a \propto b^{-2}$, which he appears to have deduced from the fact that $l \propto a^{-\frac{1}{2}}$ for a solitary wave; however, I am unable to follow his argument.) The contraction and expansion angles for the channels were roughly $6^\circ \doteqdot 0{\cdot}1$ rad, whilst the range of α was 0·18–0·50, so that (5) appears to have been marginally to well, and (7) well, satisfied; on the other hand $\alpha \ll 1$ is only marginally satisfied at the higher amplitudes. Unfortunately his results, which are reproduced in figures 1–3, do not appear to distinguish definitely between Green's law and the present prediction, $a \propto b^{-\frac{2}{3}}$.

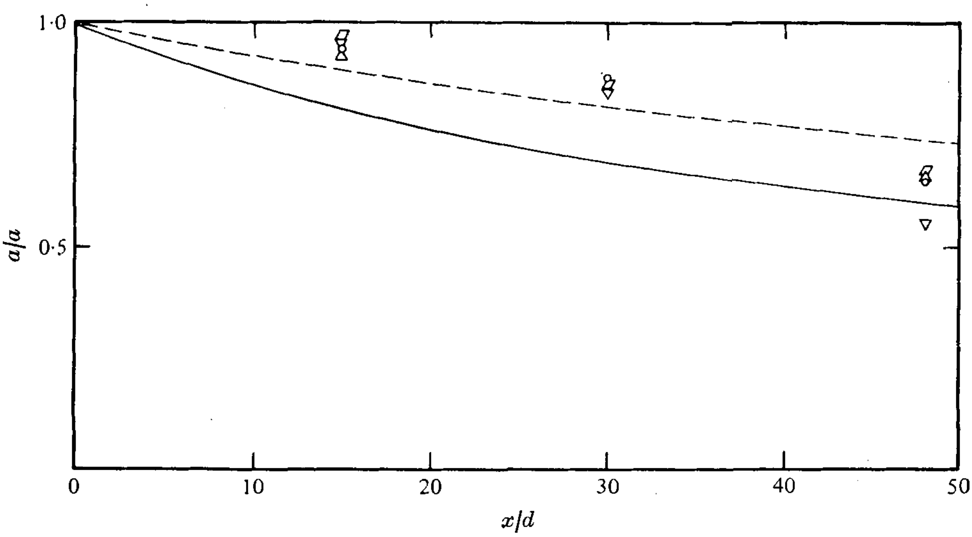

FIGURE 3. Amplitude of a solitary wave in a diverging channel for which b/d increases linearly from 5·4 to 12 as x/d increases from 0 to 50. ——, $a/a_0 = (b/b_0)^{-\frac{2}{3}}$; – – –, $a/a_0 = (b/b_0)^{-\frac{1}{2}}$. Observed data: ∇, $\alpha = a_0/d = 0{\cdot}20$; $\square$, $\alpha = 0{\cdot}35$; Δ, $\alpha = 0{\cdot}45$; $\circ$, $\alpha = 0{\cdot}50$.

The agreement between this prediction and the data for the converging channels (figures 1 and 2) appears to be fairly satisfactory (and better than that for Green's law, at least for the smaller amplitudes) if allowance is made for partial breaking at high amplitudes (as observed by Perroud) and for wall friction when the breadth of the channel is comparable with the depth. The agreement for the diverging channel (figure 3) is not satisfactory for reasons that are not clear (to me), although Perroud does suggest rather obliquely that his measurements are less accurate for smaller amplitudes (such as necessarily occur in the diverging channel). If a does vary less rapidly than $b^{-\frac{2}{3}}$, the energy argument would suggest that l does not vary like $a^{-\frac{1}{2}}$, i.e. that the solitary wave does not retain the Boussinesq profile.

This work was partially supported by the Physical Oceanography Division, National Science Foundation (NSF Grant OCE74-23791) and by the Office of Naval Research (Contract N00014-76-C-0025).

REFERENCES

JOHNSON, R. S. 1973 On the asymptotic solution of the Korteweg–de Vries equation with slowly varying coefficients. *J. Fluid Mech.* **60**, 813–824.

LAMB, H. 1932 *Hydrodynamics.* Cambridge University Press.

MADSEN, O. S. & MEI, C. C. 1969 The transformation of a solitary wave over an uneven bottom. *J. Fluid Mech.* **39**, 781–791.

MILES, J. W. 1977*a* Obliquely interacting solitary waves. *J. Fluid Mech.* **79**, 157–170.

MILES, J. W. 1977*b* Resonantly interacting solitary waves. *J. Fluid Mech.* **79**, 171–180.

PERROUD, P. H. 1957 The solitary wave reflection along a straight vertical wall at oblique incidence. Ph.D. thesis, University of California, Berkeley.